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Josef Bigün
Gösta Granlund

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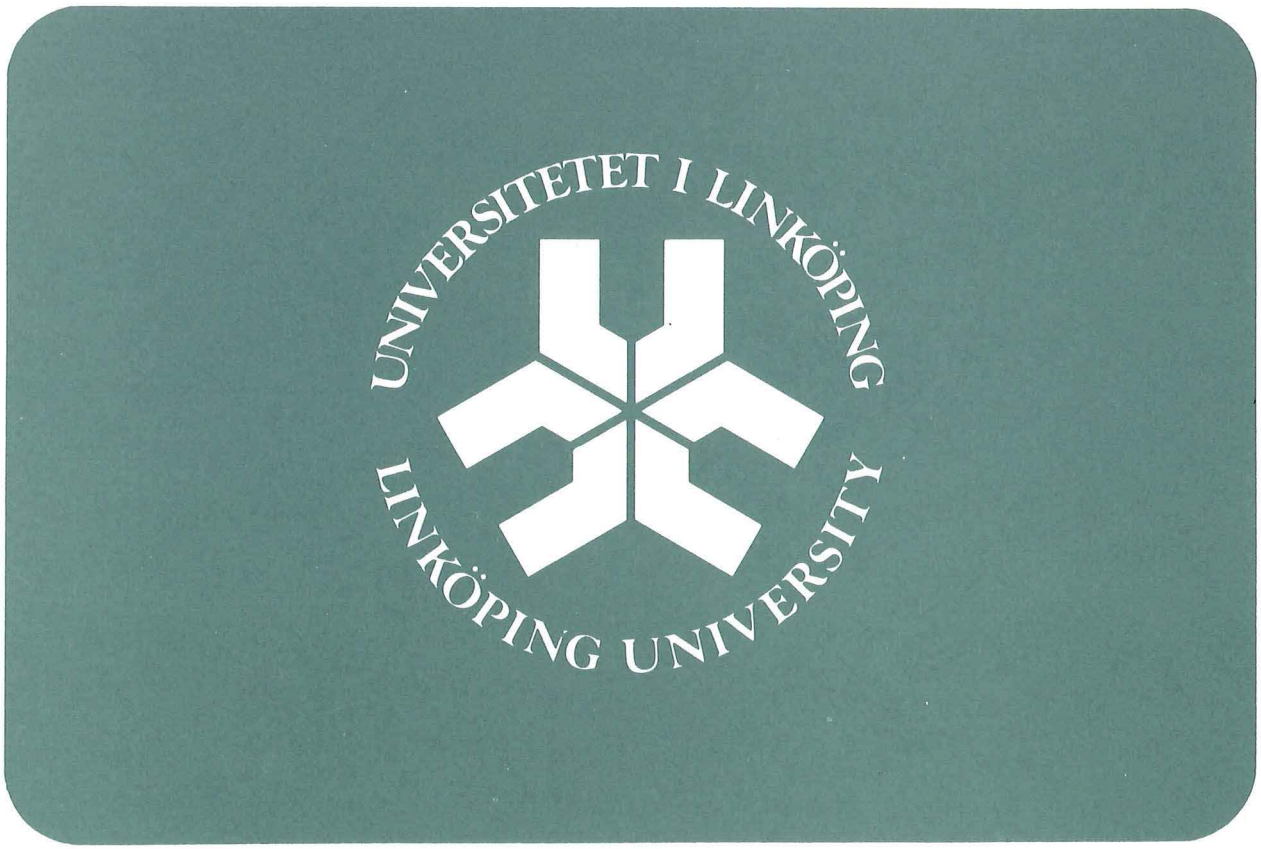

Department of Electrical Engineering
Linköping University
S-581 83 Linköping, Sweden

Linköpings tekniska högskola
Institutionen för Systemteknik
581 83 Linköping

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CENTRAL SYMMETRY MODELING

Josef Bigün, Gösta Granlund

Linköping University. Department of Electrical Engineering. S–581 83 Linköping, Sweden.

A definition of central symmetry for local neighborhoods of 2-D images is given. A complete ON-set of centrally symmetric basis functions is proposed. The local neighborhoods are expanded in this basis. The behavior of coefficient spectrum obtained by this expansion is proposed to be the foundation of central symmetry parameters of the neighborhoods. Specifically examination of two such behaviors are proposed: Point concentration and line concentration of the energy spectrum. Moreover, the study of these types of behaviors of the spectrum are shown to be possible to do in the spatial domain.

INTRODUCTION

There is a long list of operators that detect the existence of linear symmetry in a local neighborhood. Most of them measure linear symmetry in the sense of lines and edges. But there is very little done to model central symmetry. Perhaps it is because images of objects in nature, are usually more irregular than circles. Nevertheless, we believe that this is one of the symmetries which human beings utilize in early vision. It seems that central symmetry should be an additional symmetry model. The fact that circularly symmetric shapes like rotating fans, diverging rays, circularly propagating water waves... e.t.c. are observed as phosphenes when low frequency magnetic fields are applied to the temples of a subject, [1], [2], supports this belief. Moreover many manufactured objects are locally concentrated and have closed rounded boundaries. Many natural objects in low resolution images may exhibit this property like cells seen under a microscope. Conceivable application areas are object counting, classification as well as image coding and enhancement for certain types of images, possessing local central symmetry property. But first we should have an intuitive feeling about what kind of patterns are called centrally symmetric in our terminology, since it is otherwise quite a vague concept.

DEFINITION: We will call local neighborhoods centrally symmetric if the locus of iso-gray values constitute parallel lines in local polar coordinates:

$$\varphi = k_1 r + k_2 \qquad r \geq 0$$

for some constants k_1 and k_2. if the locus of iso-gray values are not curves, that is when they are regions then the borders of these regions are considered as locus. We will assume that the boundary of the neighborhood is a circle, and the origin of coordinates is the center of this circle.

DEFINITION: $C(\Omega)$ is the space of complex valued functions which are continuous on Ω except on a subset of Ω with zero measure. Ω is a circle with the radius R.

DEFINITION: (f,g) is the scalar product for $f,g \in C(\Omega)$ with

$$(f,g) \triangleq \frac{1}{|\Omega|}\int_{\Omega}\frac{1}{r}f^{*}(\bar{r})g(\bar{r})d\Omega$$

with $r = |\bar{r}|$ and:

$$|\Omega| = \int_{\Omega}\frac{1}{r}d\Omega.$$

Consider the functions, see Figure 1),

$$\Psi_{mn}(\bar{r}) = e^{i(m\omega r + n\varphi)} \tag{1}$$

with $w = \frac{2\pi}{R}$ and $m,n \in Z$. $C(\Omega)$ is a Hilbert space with the following scalar product

$$\frac{1}{2\pi R}\int_{0}^{2\pi}\int_{0}^{R}\bar{f}(r,\varphi)g(r,\varphi)drd\varphi.$$

Moreover $\{\Psi_{mn}\}_{m,n\in Z}$ is dense in $C(\Omega)$, which follows from the Fourier series expansion theory on a rectangle, [3]. But this scalar product is the same scalar product defined earlier with, Ω, being a circle. By that we have established that $C(\Omega)$ is a Hilbert space with the scalar product given in the definition. Now let us consider the neighborhood Ω, around an examined point in an image. Assume that the polar coordinates, $r = |\bar{r}|$ and $\varphi = arg(\bar{r})$, referred to in the following are relative to the examined point, and the positive x axis from the examined point.

Let the real function $f(\bar{r})$ express the gray values in Ω, with the center at the examined point, such that $\bar{r}$ is the local coordinate vector. Then one can expand f as

$$f(\bar{r}) = \sum_{m,n\in Z} c_{mn}\Psi_{mn}(\bar{r}) \tag{2}$$

with

$$\begin{aligned} c_{mn} &= (f,\Psi_{mn}) \\ &= \frac{1}{2\pi R}\int_{\Omega}\frac{1}{r}f(\bar{r})e^{i(m\omega r+n\varphi)}d\Omega \end{aligned} \tag{3}$$

because $C(\Omega)$ is a Hilbert space and $\{\Psi_{mn}\}_{m,n\in Z}$ constitutes a complete orthonormal base:

$$(\Psi_{mn},\Psi_{m'n'}) = \delta_{mm'}\delta_{nn'} \tag{4}$$

with $\delta_{mm'}$ being the usual Kronecker delta.

POINT CONCENTRATION

DEFINITION: Let P be an operator from $C(\Omega)$ to the function set X, $X \subset C(\Omega)$. Then P is a projection from $C(\Omega)$ to X if

$$P^2 f = Pf$$

for all f in $C(\Omega)$.

Our goal is to find an algorithm based on operations done in the spatial domain which still gives some indication about whether the energy is concentated to a point in the frequency domain. The algorithm should posses the following properties:

1) Whenever the neighborhood, $f(\bar{r})$, is equivalent to one of the basis functions, Ψ_{mn}, except possibly for a scale factor B, the algorithm should detect this particular basis function save a sign change of it's index tuple, (n,m). That is $(f,\Psi_{m'n'}) = 0$ for all tuples (n',m') except for a tuple (n,m). In other cases it should give some sort of dominating tuple (n,m). It should be noted that Ψ_{mn} is complex valued. For real neighborhoods consisting of the real or imaginary part of a Ψ_{mn}, this condition will be enough to identify the neighborhood except possibly a phase factor. Given the tuple (n,m), Ψ_{mn} is unique. Call the operator of finding the tuple (n,m), and associating the function Ψ_{mn} to that, as P then:

$$P^2 f = Pf = \Psi_{mn}$$

for any $f \in C(\Omega)$. This is equivalent to saying that the sought algorithm is a projection to the countable set $\{\Psi_{mn}\}$, according to the projection definition above.

2) The projection value (or parameter) should be rotation and radial phase invariant for pure inputs of:

$$f(r,\varphi) = B\Psi_{mn}(r,\varphi)$$

with some scalar B. That is

$$Pf(r+r_0,\varphi+\varphi_0) = Pf(r,\varphi) = \Psi_{mn}$$

should be fullfilled.

3) Whenever the spectrum of the real valued local neighborhood differs from a pattern with a point concentrated spectrum, an uncertainty parameter should reflect that. By attaining low values, for example, this parameter could indicate the relevance of the projection parameter, and conversely to suppress it if the neighborhood differs from a central symmetric pattern.

The use of the uncertainty parameters is indicated in [4]. The uncertainty parameter and the projection parameter are combined in every point of the image to form a vector, in such a way that the magnitude of this vector becomes inverse proportional to uncertainty parameter, (the confidence in the projection parameter) and the argument of it becomes the projection parameter. This can be visualised by allowing the magnitude to modulate the intensity of a point in a color TV monitor and the argument of it, representing the projection parameter, to modulate the color of the same point. The result is a color image representing a decision in every neighborhood of the original image. The projection parameter and the confidence parameter values evaluated in every point in the image can be thought of as two separate images influencing each other. A point

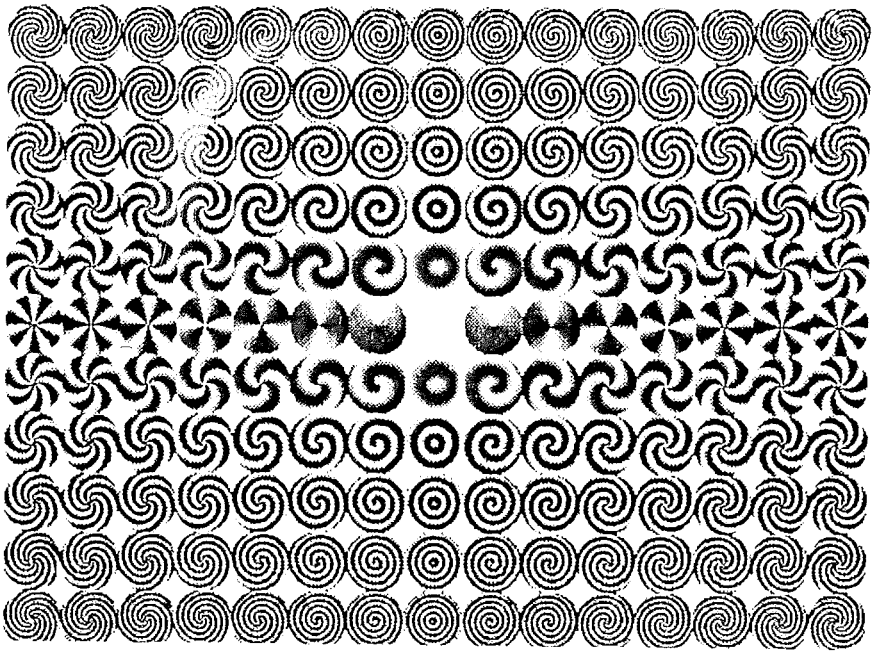

Figure 1) The image illustrates some of the basis functions Ψ_{mn}. The real parts of Ψ_{mn} are mapped linearly to the gray values of the monitor.

with a low confidence level looks dark in the resulting image, no matter what the color of the point is. A point with a high confidence level is emphasized by illumination, and it's color is revealed.

The algorithm we propose consists of the computations described by (5)-(9):

$$A \triangleq \|f\| \tag{5}$$

$$m_d \triangleq \frac{\|D_r f\|}{A\omega} \tag{6}$$

$$n_d \triangleq \frac{\|D_\varphi f\|}{A} \tag{7}$$

$$C_{\Omega m}^2 \triangleq \frac{\|D_r^2 f\|^2}{A^2\omega^4} - m_d^4 \tag{8}$$

$$C_{\Omega n}^2 \triangleq \frac{\|D_\varphi^2 f\|^2}{A^2} - n_d^4 \tag{9}$$

m_d and n_d are radial respectively angular frequency measures. $C_{\Omega m}$ and $C_{\Omega n}$ are the uncertainty measures associated with m_d respectively n_d. Denote the projection parameters by the tuple $(\hat{n}, \hat{m})$. That is $(\hat{n}, \hat{m})$ points out the location of an eventual point concentration in the spectrum. To produce $\hat{m}$ and $\hat{n}$ from m_d and n_d, we observe that $\hat{m}$ and $\hat{n}$ should be integers. Moreover they take positive as well as negative values. However since we assume real valued images, the requested point concentration will consist of two concentrations symmetrically located around the origin of the coordinates in the spectrum. This is due to the Hermitian property of the coefficient transformation. Hence we need only give the position of one of these concentrations. Thus we can assume

that $\hat{m}$ is allways positive. We will simply assign to $\hat{m}$ and $\hat{n}$ the closest integers to m_d and n_d with proper sign:

$$\begin{aligned} \hat{m} &= round(m_d) \\ \hat{n} &= sign \times round(n_d) \qquad sign \in \{-1, 1\} \end{aligned} \tag{10}$$

sign is the sign of $\sum_{m,n\in Z} mn\frac{|c_{mn}|^2}{A^2}$, the calculation of which is given in next section. Let us see what (5)-(9) does for a neighborhood :

$$f = \sum_{m,n\in Z} c_{mn}\Psi_{mn} \tag{11}$$

We get through (5)

$$A^2 = \sum_{m,n\in Z} |c_{mn}|^2$$

This is the energy of the neighborhood in terms of the centrally symmetric function set $\{\Psi_{mn}\}$. (4) together with (6), (11) yields:

$$\begin{aligned} m_d = \frac{\|D_r f\|}{A\omega} &= \| \sum_{m,n\in Z} i\frac{mc_{mn}}{A}\Psi_{mn}\| \\ &= (\sum_{m,n\in Z} \frac{c_{mn}^2}{A^2}m^2)^{\frac{1}{2}} \end{aligned} \tag{12}$$

Hence m_d is the weighted root mean square of all radial frequency measures, m. It should be observed that a particular radial frequency number, m, is weighted by the uniform sum of all angular frequency energies. The weights constitute energy distribution of the input function. The higher the energy share of Ψ_{mn} in the total energy, the more m_d will be close to m. (12) fulfills obviously the projection requirement after rounding m_d to the closest integer, $\hat{m}$. Similarly n_d will be the weighted mean square of all angular frequencies of different order:

$$n_d = (\sum_{m,n\in Z} \frac{|c_{mn}|^2}{A^2}n^2)^{\frac{1}{2}} \tag{13}$$

The latter is insensitive to the sign changes in n. The consequence of this is a real neighborhood of

$$f = \Psi_{mn} + \Psi_{-m-n} + \Psi_{m-n} + \Psi_{-mn},$$

is projected to a $\Psi_{|m||n|}$ input. The decision is to the favor of one of the two equally strong candidates. When $f = \Psi_{mn} + \Psi_{-m-n}$ then $m_d = |m|$ and $n_d = |n|$ which in turn reflects the necessity of the variable *sign* referred to earlier (10). Uncertainty

parameters $C_{\Omega m}$ and $C_{\Omega n}$ are proposed to be as in (8) and (9), and $C_{\Omega m}$ yield through (4), (8), (11), (12)

$$C_{\Omega m}^2 = \frac{\|D_r^2 f\|^2}{A^2\omega^4} - m_d^4 = \sum_{m,n\in Z} \frac{|c_{mn}|^2}{A^2} m^4$$
$$- 2 \sum_{m,n\in Z} \frac{|c_{mn}|^2}{A^2} m^2 m_d^2 + m_d^4 \sum_{m,n\in Z} \frac{|c_{mn}|^2}{A^2} \tag{14}$$
$$= \sum_{m,n\in Z} \frac{|c_{mn}|^2}{A^2} (m^2 - m_d^2)^2$$

which can be viewed as a weighted variance for the integers m^2. It attains it's minimum in the case when

$$\frac{|c_{mn}|^2}{A^2}(m^2 - m_d^2)^2 = 0$$

for all $n, m \in Z$. This occurs if and only if

$$\sum_n \frac{|c_{mn}|^2}{A^2} = 1$$

for some $m = m'$ since m_d^2 is constant. Thus if $C_{\Omega m}$ is zero then there exists one unique radial frequency in the neighborhood. And it is given by the estimation, m_d. When this is the case the energy is concentrated to a horizontal line through $m = m_d$. Since $C_{\Omega m}$ is a variance it also reveals some information about the shape of spectral density of the neighborhood. If $C_{\Omega m}$ is small then it is likely to think that the neighborhood is a degraded version of a wave with a well defined radial frequency, m_d. Conversely it is unlikely that the association of m_d to the neighborhood will be relevant, if $C_{\Omega m}$ is large. Interpretations of n_d and $C_{\Omega n}$ are similar to m_d and $C_{\Omega m}$'s. Given m_d, n_d, and the *sign* parameters the tuple $(\hat{n}, \hat{m})$ is computed according to (10). We adopt the uncertainty parameters $C_{\Omega n}$ and $C_{\Omega m}$ for $\hat{n}$ respectively $\hat{m}$. We propose $C_{\Omega\nu}$,

$$C_{\Omega\nu}^2 = C_{\Omega m}^2 + C_{\Omega n}^2 \tag{15}$$

to be the uncertainty parameter for the tuple $(\hat{n}, \hat{m})$. $C_{\Omega\nu} = 0$ if and only if $C_{\Omega m} = C_{\Omega n} = 0$. But $C_{\Omega m} = 0$ if and only if the total energy is concentrated on a horizontal line and $C_{\Omega n} = 0$ if and only if the total energy is concentrated on a vertical line. The only possibility for the neighborhood to fulfill these two requirements is being an input possessing total point concentration in it's spectrum, with location on the intersection of the lines $m = m_d$, $n = n_d$.

LINE CONCENTRATION

We will examine whether the energy spectrum has line concentration. Let the line we look for be

$$m = \tan(\theta) n \tag{16}$$

we assume that the line goes through the origin of the coordinates in the coefficient domain. Since the real functions coefficient transforms should be Hermitian, their energy spectra are even, forcing an eventuall line concentration to pass through the origin of the coordinates of the coefficient plane. A real neighborhood f can be expanded in the basis functions as before, yielding:

$$f = \sum_{m,n \in Z} c_{mn} \Psi_{mn}$$

with the Hermitian coefficients c_{mn}. The energy concentration of the neighborhoods in general degrades from a line through the origin of the coordinates. Let us measure this degradation by $C_{\Omega\theta}$, which is the average sum of the squares of the distances of the spectrum points to the line given by (16):

$$\begin{aligned} C_{\Omega\theta} &\triangleq \sum_{m,n \in Z} (m - \tan(\theta)n)^2 \cos^2(\theta) \frac{|c_{mn}|^2}{A^2} \\ &= \sin^2(\theta) \sum_{m,n \in Z} n^2 \frac{|c_{mn}|^2}{A^2} + \cos^2(\theta) \sum_{m,n \in Z} m^2 \frac{|c_{mn}|^2}{A^2} \\ &- \sin(2\theta) \sum_{m,n \in Z} mn \frac{|c_{mn}|^2}{A^2} \end{aligned} \tag{17}$$

We want to find a θ which minimizes $C_{\Omega\theta}$. This is the least square estimation of θ and it is straight forward to find θ:

$$\frac{dC_{\Omega\theta}}{d\theta} = (n_d^2 - m_d^2)\sin(2\theta) - 2p\cos(2\theta) \tag{18}$$

where

$$p = \sum_{m,n \in Z} mn \frac{|c_{mn}|^2}{A^2}$$

If $n_d^2 - m_d^2 \neq 0$ or $p \neq 0$ then choose the minimizing θ as:

$$\theta_d = \frac{1}{2}\tan^{-1}(n_d^2 - m_d^2, 2p). \tag{19}$$

The degradation or uncertainty measure is given by substituting (19) in (18) and using the trigonometric half angle formulas:

$$C_{\Omega\theta} = \frac{1}{2}\left(n_d^2 + m_d^2 - \sqrt{(n_d^2 - m_d^2)^2 + 4p^2}\right). \tag{20}$$

The angle given by (19) gives the axis around which the moment of inertia is minimum and the moment of inertia is given by (20) if $\frac{|c_{mn}|^2}{A^2}$ is seen as a point mass, [5]. The omitted case when both $p = 0$ and $n_d^2 - m_d^2 = 0$ corresponds to local neighborhoods with

no specific orientation. Because $\frac{dC_{\Omega\theta}}{d\theta}$ vanishes according to (16), any θ would work as minimizing argument to (17). This case implies that

$$\sum_{m,n\in Z} m^2 \frac{|c_{mn}|^2}{A^2} = \sum_{m,n\in Z} n^2 \frac{|c_{mn}|^2}{A^2}$$

$$\sum_{m,n\in Z} mn \frac{|c_{mn}|^2}{A^2} = 0$$

The class of functions having this property in their spectra is the class of functions with coinciding principal axes in the coefficient domain. Neighborhoods of Ψ_{00}, $\Psi_{mn} + \Psi_{-m-n} + \Psi_{m-n} + \Psi_{-mn}$ are examples of such functions, we observe that $m_d^2 = 0$ and $n_d^2 = 0$ implies that the neghborhood is a constant function and consequently have no orientation. Thus to keep the consistency of the meaning of $C_{\Omega\theta}$ in the case when $n_d^2 - m_d^2 = p = 0$, we should define $C_{\Omega\theta} = \infty$ and leave θ undefined. p which is needed to calculate θ_d and $C_{\Omega\theta}$ according to (19) and (20), can be easily found in the spatial domain to be:

$$p = \sum_{m,n\in Z} mn \frac{|c_{mn}|^2}{A^2} = \frac{1}{A^2\omega}\left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \varphi}\right)$$

Implementation of the scalar products given above for every neighborhood of a digitized image is straight forward after the usage of the chain rules:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos(\varphi) + \frac{\partial f}{\partial y}\sin(\varphi)$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial y} r\cos(\varphi) - \frac{\partial f}{\partial x} r\sin(\varphi)$$

By that we can transfer the scalar products to be valid for functions defined in cartesian coordinates. At this point we can use either the band limited signal theory or some quadrature rule to evaluate the resulting integrals, given that we know $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at a rectangular net of points. It can be shown that the scalar product evaluations at every point is obtained by convolutions with FIR-filters.

CONCLUSION

Both the point concentration parameters m_d, and n_d and the line concentration parameter θ_d are best fits of a point or a line respectively through the origin of coordinates of the coefficient domain. The best fit is in the sense that the two variance measures given, which are adopted as uncertainty measures, are minimized. It is interesting to note that the approach lends itself to linear symmetry parameter extraction as well, with a minor change. By linear symmetry we mean the neighborhoods with iso–gray values being straight lines in cartesian coordinates. Parallel lines belong to such neighborhoods. Hence it is possible to find the dominating frequency and the dominating orientation of a neighborhood, [6], with the least error variance in the fourier domain in a similar manner. The only difference is the scalar product and the shape of the neighborhood. The

scalar product of the linear symmetry case becomes the usual $\mathcal{L}^2(\Omega)$ scalar product with Ω being a rectangle. The complete ON-basis set is of course $\{e^{i(m\omega_x x+n\omega_y y)}\}_{m,n\in Z}$.

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